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Harmonic superspaces of extended supersymmetry: I. The calculus of harmonic variables

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Abstract. The main technical apparatus of the harmonic superspace approach to extended SUSY, the calculus of harmonic variables on homogeneous spaces of the SUSY automorphism groups is presented in detail for $N=2,3,4$. We construct the basic harmonics for the coset manifolds G/H with G = SU(2), H = U(1); G = SU(3), H = SU(2) × U(1) and H = U(1) × U(1); G = SU(4), H = SU(3) × U(1), H = SU(2) × SU(2) × U(1), H = SU(2) × U(1) × U(1) and H = U(1) × U(1) × U(1), G = USp(2), H = SU(2) × SU(2), H = SU(2) × U(1) and H = U(1) × U(1) and tabulate a number of useful relations between them.

1. Introduction

Recently, the concept of harmonic superspace was proposed for obtaining an unconstrained formulation of $N=2$ matter, super Yang-Mills (SYM) and supergravity theories (Galperin *et al* 1984). The same approach proved to be suitable for constructing an off-shell $N=3$ SYM theory (Galperin *et al* 1985), thus circumventing the famous $N=3$ barrier (Roček and Siegel 1981, Rivelles and Taylor 1982). The main idea consists of enlarging the ordinary superspace by some new even coordinates u_i^A that form a basis set of harmonic functions on some coset manifold G/H, G being the group of automorphisms of the supersymmetry (SUSY) algebra (i.e. SU(2) in $N=2$ and SU(3) in $N=3$ cases, etc), H being one of its subgroups. Then it is possible to extract from this enlarged superspace a subspace, called analytic (Galperin *et al* 1984, 1985), relevant for constructing unconstrained SUSY theories. The fundamental superfield objects of those theories appear very elegantly as analytic functions defined on this subspace.

The purpose of the present paper is to give a kind of glossary of harmonic calculus for the simplest groups and their cosets G/H. Only a few of the examples considered here have been already used in constructing extended SUSY theories: the relevance of the remaining ones may be revealed later. There is a remarkable intimate connection between the geometric structure of a SUSY theory and the choice of the homogeneous space G/H used to define the harmonic u_i^A. We intend to list all harmonic superspaces of interest and their analytic subspaces in another paper based on the matter given here.

We would like to emphasise that for the moment we are not ready to discuss a realistic theory of the $N=4$ SYM in harmonic superspace. There appear some difficulties for this theory, especially the one associated with the reality constraint on the field strength (Ahmed *et al* 1985). There are some radical ways to resolve this problem, but the discussion of them is beyond the scope of the present paper. We only note that these essentially use the harmonic calculus on SU(4) given here.

The paper is organised as follows. Section 2 describes the general techniques of constructing harmonics on some coset space G/H. For pedagogical reasons we illustrate these techniques by the familiar SU(2)/U(1) example extensively used in $N=2$ SUSY (Galperin *et al* 1984). All other cases are treated analogously to this simplest one. Section 3 treats cosets associated with G = SU(3) (both for H = SU(2) × U(1) and U(1) × U(1)), in § 4 we consider cosets of G = SU(4) for H = SU(3) × U(1), SU(2) × SU(2) × U(1), SU(2) × U(1) × U(1) and U(1) × U(1) × U(1). Section 5 is devoted to the case of G = USp(2) with H = SU(2) × SU(2), SU(2) × U(1) and U(1) × U(1). Some brief concluding remarks are given in § 6.

2. General techniques and SU(2)/U(1) example

We begin by introducing the set of harmonics u_i^A defined on the manifold G/H, H ⊂ G; H, G being compact groups. We take the matrix representation of G and H, and so we can define $u_i^j \in \mathrm{G/H}$, to be, say, $N\times N$ matrix. For instance, if G = SU(2) and H = U(1) is its diagonal subgroup, then

$$u_i^j = [\exp \mathrm{i}(\varphi T^{++} + \bar{\varphi} T^{--})]_j^i \tag{2.1}$$

where φ is complex variable,

$$T^{++} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T^{--} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are SU(2) generators, $i, j = 1, 2$.

The action of an element g_i^k of G on the coset element is defined as (Salam and Strathdee 1982)

$$u_i^j \to u_i'^j = g_i^k u_k{}^l h_l^j(g, u) \tag{2.2}$$

where $h \in \mathrm{H}$ is a 'compensating' right H transformation. Next, let us introduce a set of basis vectors q_i^A in the G representation space (normally, the fundamental representation is considered) such that for any $h \in \mathrm{H}$

$$q_i^A \to q_i'^A = h_i^l q_l^A \equiv q_i^B h_{\mathrm{B}}^A \tag{2.3}$$

and h_{B}^A has a block-diagonal form. Here indices i, j refer to the fundamental irrep of G and $A, B, \ldots$ run through all irreps of H which are contained in the fundamental irrep of G. Thus h_{B}^A is the matrix in this reducible H representation.

Now, the harmonics are defined as

$$\begin{aligned} &u_i^A = u_i^j q_j^A \\ &\mathrm{G}: u_i^A \to u_i'^A = g_i^k u_k^B h_{\mathrm{B}}^A(g, u) \end{aligned} \tag{2.4}$$

and they belong to the representation space of G × representation space of H. So, they transform under G × H, where G acts from the left and H from the right.

According to (2.2) and (2.3) the right H transformation is not independent, it is completely fixed by the left G ones. However, we can in fact introduce u_i^A as 'free' objects in the G × H representation space, i.e. as a kind of vielbein converting G reps into H reps. The group G is originally realised on them by left multiplications (without compensating H transformations). At the same time, u_i^A are transformed from the right by a new independent gauge group H whose parameters are arbitrary functions of u_i^A themselves. If we then fix the H gauge so as to reduce the number of independent parameters in u_i^A (equal originally to dim G) to dim G/H, we recover the standard coset formulation (2.4). Thus, requiring invariance under right gauge H transformations one may adhere to the 'vielbein' interpretation of u_i^A which is convenient in a number of aspects.

For the SU(2)/U(1) case the q_i^A vectors are evidently

$$q_i^+ = \begin{pmatrix}1\\0\end{pmatrix}, \qquad q_i^- = \begin{pmatrix}0\\1\end{pmatrix}$$

(+ and − label the U(1) irreps) that according to (2.3) and (2.4), gives us the harmonics from (Galperin *et al* 1984), namely

$$\begin{aligned} u_i^+ &= u_i^1 \\ u_i^- &= u_i^2 \end{aligned} \qquad u_i^j = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix}. \tag{2.5}$$

Let us point out that, since u_i^j is a G matrix, this implies some relations between u_i^A, e.g. in the SU(2)/U(1) case:

$$u^\dagger u = uu^\dagger = \mathbb{1}_2 \tag{2.6a}$$

whence

$$u^{+i}u_i^- = 1 \Leftrightarrow u_i^+u_j^- - u_j^+u_i^- = \varepsilon_{ij} \tag{2.6b}$$

$$u^{+i}u_i^+ = u^{-i}u_i^- = 0 \tag{2.6c}$$

$$u_i^- = \overline{(u^{+i})}.$$

Here (2.6) represents the unitarity and unimodularity properties of SU(2) matrices. We shall see that such constraints are prototypes of those for the more complicated cases listed below. Moreover, (2.6) gives us the possibility to convert, as has been stated above for the general case, SU(2) indices into U(1) ones and vice versa, namely

$$\begin{aligned} \psi_i &= u_i^+\psi^- - u_i^-\psi^+ \\ \psi^\pm &= u_i^\pm \psi^i, \qquad \psi^j = \varepsilon^{ji}\psi_i \\ u^{+j} &= \varepsilon^{ji}u_i^+. \end{aligned} \tag{2.6$'$}$$

Finally we note that with the help of harmonics u_i^A one can expand a function defined on G/H and belonging in external indices to an H representation, in powers of u. Namely

$$F^{(AB\ldots C)}_{(\mathrm{G/H})} = \sum u_i^A u_j^B \ldots u_l^C f^{ij\ldots l} \tag{2.7}$$

where $f^{ij\ldots l}$ are G irrep coefficients independent of u and the summation is over all (usually infinitely many) monomials in u_i^A belonging to the same H representation as F (these are nothing but higher harmonics on G/H). For example, in the SU(2)/U(1) case one has

$$F^{(n+)}_{(\mathrm{SU(2)/U(1)})} = \sum_{k=0}^{\infty} u_{i_1}^+ \ldots u_{i_{k+n}}^+ u_{j_1}^- \ldots u_{j_k}^- f^{(i_1\ldots j_k)}. \tag{2.8}$$

For such functions (they are exactly the reps of G induced from H irreps) one may define covariant differentiation with respect to the coset parameters using the standard technique of Cartan's forms (see, e.g., Salam and Strathdee 1982). We would not present a general formula but just illustrate this again by the example of SU(2)/U(1). The covariant derivatives in $--$ and $++$ directions of SU(2)/U(1) have a very simple form in terms of harmonics $u_i^\pm$:

$$D^{++} = u_i^+ \partial/\partial u_i^-, \qquad D^{--} = u_i^- \partial/\partial u_i^+. \tag{2.9}$$

Together with the operator

$$D^3 = \tfrac{1}{2}(u^{+i} \partial/\partial u^{+i} - u^{-i} \partial/\partial u^{-i})$$

(which is just the generator of right U(1) transformations and is equal to overall U(1) charge when applied to any function of the type (2.8)) they constitute an SU(2) algebra:

$$[D^{++}, D^{--}] = 2D^3, \qquad [D^{++}, D^3] = -D^{++}, \qquad [D^{--}, D^3] = D^{--}.$$

The last property can be understood from the fact that D^{++}, D^{--}, D^3 can be alternatively defined as generators of right SU(2) transformations of the coset SU(2)/U(1) (which are realised on indices +, − of harmonics). One more remark concerning that case is in order. Besides the usual complex conjugation (−)

$$u^{\pm i} \xrightarrow{(-)} \overline{u^{\pm i}} = \pm u_i^\mp \tag{2.10}$$

one can define another involution (*)

$$u_i^\pm \xrightarrow{(*)} (u_i^\pm)^* = \pm u_i^\mp \tag{2.11}$$

allowing, together with (2.10), to define self-conjugated charged objects, say

$$F^{(n+)} = \overline{(F^{(n+)})}^{\,*} \qquad (n = 2k). \tag{2.12}$$

The geometric meaning of * is very simple: it takes any point of the sphere SU(2)/U(1) to the opposite one, i.e. it is the antipodal mapping of this sphere. We shall see that such an operation is not always possible and correspondingly the reality in the sense of (2.12) can be defined only for certain G/H. This places a strong restriction on the choice of subgroup H.

3. The harmonics for G = SU(3)

Now we are ready to collect useful formulae for harmonics of G/H, G = SU(3), H = SU(2) × U(1) and H = U(1) × U(1) (see table 1).

Table 1.

G = SU(3)	H = SU(2) × U(1)		H = U(1) × U(1)	
Generators of H in the 3 × 3 matrix form	SU(2)	$T_i = \begin{pmatrix} \tau_i & 0 \\ 0 & 0 \end{pmatrix}$ where τ_i are 2 × 2 SU(2) generators	$U_1(1)$	$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	U(1)	$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$U_2(1)$	$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Table 1. (continued)

G = SU(3)	H = SU(2) × U(1)	H = U(1) × U(1)
q_i^A	SU(2) doublet $q_i^{+a} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ SU(2) singlet $q_i^- = \begin{pmatrix}0\\0\\1\end{pmatrix}$	$q_i^{(1,1)} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $q_i^{(0,-2)} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $q_i^{(-1,1)} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ in what follows, a couple of indices (a, b) represents T_1 and T_2 charges, respectively
Harmonics	u_i^{+a}, u_i^{--} and their conjugates u_a^{-i}, u^{++i}	$u_i^{(1,1)}, u_i^{(-1,1)}, u_i^{(0,-2)}$ and their conjugates $u^{(-1,-1)i}, u^{(1,-1)i}, u^{(0,2)i}$
Unitarity $u^+u = uu^+ = 1$	$u^+_{ia}u_b^{-i} = \varepsilon_{ab}$ $u_i^{--}u^{++i} = 1$ $u^+_{ia}u^{++i} = u_b^{-i}u_i^{--} = 0$ $-\varepsilon^{ab}u^+_{ai}u_b^{-j} + u_i^{--}u^{++j} = \delta_i^j$	$u_i^{(1,1)}u^{(-1,-1)i} = 1$ $u_i^{(-1,1)}u^{(1,-1)i} = 1$ $u_i^{(0,-2)}u^{(0,2)i} = 0$ $u_i^{(1,1)}u^{(-1,-1)j} + u_i^{(-1,1)}u^{(1,-1)j}$ $+ u_i^{(0,-2)}u^{(0,2)j} = \delta_i^j$
Unimodularity condition $\det u = 1$	$\varepsilon^{ab}u^+_{ai}u^+_{bj} = \varepsilon_{ijk}u^{++k}$ $\varepsilon^{ijk}u^+_{ai}u^+_{bj}u_k^{--} = \varepsilon_{ab}$ and their conjugates	$u_i^{(0,-2)} = \varepsilon_{ijk}u^{(-1,-1)j}u^{(1,-1)k}$ and their conjugates
Harmonic derivatives preserving unitarity and unimodularity	$D_a^{3+} = u^{++i}\,\partial/\partial u^{-ai} + u^+_{ai}\,\partial/\partial u_i^{--}$ $D_b^{3-} = u_i^{--}\,\partial/\partial u_i^{+b} + u_b^{-i}\,\partial/\partial u^{++i}$	$D^{(1,3)} = -u^{(0,2)i}\,\partial/\partial u^{(-1,-1)i}$ $+ u_i^{(1,1)}\,\partial/\partial u_i^{(0,-2)}$ $D^{(-1,3)} = -u^{(0,2)i}\,\partial/\partial u^{(1,-1)i}$ $+ u_i^{(-1,1)}\,\partial/\partial u_i^{(0,-2)}$ $D^{(2,0)} = -u^{(1,-1)i}\,\partial/\partial u^{(-1,-1)i}$ $+ u_i^{(1,1)}\,\partial/\partial u_i^{(-1,1)}$ and their conjugates $D^{(1,-3)} = \bar{D}^{(-1,3)}$ $D^{(-1,-3)} = \bar{D}^{(1,3)}$ $D^{(-2,0)} = \bar{D}^{(2,0)}$
Converting indices	$\psi_i = -\varepsilon_{ab}u_i^{+a}\psi^{-b} + u_i^{--}\psi^{++}$ $\psi^{-b} = u^{-bj}\psi_j$ $\psi^{++} = u^{++j}\psi_j$	$\psi_i = u_i^{(1,1)}\psi^{(-1,-1)} + u_i^{(-1,1)}\psi^{(1,-1)}$ $+ u_i^{(0,-2)}\psi^{(0,2)}$ $\psi^{(\pm1,-1)} = u^{(\pm1,-1)j}\psi_j$ $\psi^{(0,2)} = u^{(0,2)j}\psi_j$
Reality and involution (*) (if exists)	no	yes $u_i^{(1,1)} \leftrightarrow u_i^{(0,-2)}$ $u_i^{(-1,1)} \leftrightarrow -u_i^{(-1,1)}$

Table 2.

G = SU(4)	H = SU(3) × U(1)	H = $SU_1(2) \times SU_2(2) \times U(1)$	H = $SU(2) \times U_1(1) \times U_2(1)$	H = $U_1(1) \times U_2(1) \times U_3(1)$
Generators of H in the 4×4 matrix form	SU(3) $T_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & 0 \end{pmatrix}$ where λ_i are 3×3 SU(3) generators U(1) $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$	$SU_1(2)$ $T_a = \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix}$ $SU_2(2)$ $T_p = \begin{pmatrix} 0 & 0 \\ 0 & \tau_p \end{pmatrix}$ U(1) $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ τ_a, τ_p are 2×2 $SU_1(2)$ and $SU_2(2)$ generators	SU(2) $T_a = \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix}$ $U_1(1)$ $T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $U_2(1)$ $T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$U_1(1)$ $T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $U_2(1)$ $T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $U_3(1)$ $T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
q_i^A	SU(3) triplet $q_i^{+a} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ SU(3) singlet $q_i^{(3-)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$SU_1(2)$ doublet $q_i^{+a} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $SU_2(2)$ doublet $q_i^{-p} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	SU(2) doublet $q_{ia}^{(0,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ SU(2) singlets $q_i^{(1,-1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $q_i^{(-1,-1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$q_i^{(1,0,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $q_i^{(-1,0,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $q_i^{(0,1,-1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $q_i^{(0,-1,-1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
Harmonics	$u_i^{a(+)}, u_i^{(3-)}$ and their conjugates $u_a^{i(-)}, u^{i(3+)}$	$u^{a(+)}, u_i^{p(-)}$ and their conjugates $u_a^{i(-)}, u_p^{i(+)}$	$u_{ia}^{(0,1)}, u_i^{(1,-1)}, u_i^{(-1,-1)}$ and their conjugates $u^{ia(0,-1)}, u^{i(-1,1)}, u^{i(1,1)}$	$u_i^{(1,0,1)}, u_i^{(-1,0,1)}, u_i^{(0,1,-1)}, u_i^{(0,-1,-1)}$ and their conjugates $u^{(-1,0,-1)i}, u^{(1,0,-1)i}, u^{(0,-1,1)i}, u^{(0,1,1)i}$

Unitarity $u^+u=uu^+=1$	$u_i^{a(+)}u_b^{i(-)}=\delta_b^a$ $u_i^{a(+)}u^{i(3-)}=u_i^{(3-)}u_a^{i(+)}=0$ $u_i^{(3-)}u^{i(3+)}=1$ $u_i^{a(+)}u_a^{j(-)}+u_i^{(3-)}u^{j(3+)}=\delta_i^j$	$u_{ia}^{(+)}u_b^{i(-)}=\varepsilon_{ab}$ $u_{ip}^{(-)}u_q^{i(+)}=\varepsilon_{pq}$ $u_{ia}^{(+)}u^{ja(-)}+u_{iq}^{(-)}u^{jq(+)}=-\delta_i^j$	$u_{ia}^{(0,1)}u_b^{i(0,-1)}=\varepsilon_{ab}$ $u_i^{(1,-1)}u^{i(-1,1)}=1$ $u_i^{(-1,-1)}u^{i(1,1)}=1$ all other SU(4) contractions vanish $-u_{ia}^{(0,1)}u^{ja(0,-1)}+u_i^{(1,-1)}u^{j(-1,1)}$ $+u_i^{(-1,-1)}u^{j(1,1)}=\delta_i^j$	$u_i^{(1,0,1)}u^{(-1,0,-1)i}=1$ $u_i^{(-1,0,1)}u^{(1,0,-1)i}=1$ $u_i^{(0,1,-1)}u^{(0,-1,1)i}=1$ $u_i^{(0,-1,-1)}u^{(0,1,1)i}=1$ all other SU(4) contractions vanish $u_i^{(1,0,1)}u^{(-1,0,-1)j}+u_i^{(1,0,-1)}u^{(-1,0,1)j}$ $+u_i^{(0,-1,-1)}u^{(0,1,1)j}+u_i^{(0,1,-1)}u^{(0,-1,1)j}$ $=\delta_i^j$
Unimodularity conditions det $u=1$	$\varepsilon^{ijkl}u_i^{a(+)}u_j^{b(+)}u_k^{(3-)}u_l^{c(+)}$ $=-\varepsilon^{abc}$	$\varepsilon^{ab}\varepsilon^{ijkl}u_{ia}^{(+)}u_{jb}^{(+)}$ $=u_p^{k(+)}u_q^{l(+)}\varepsilon^{pq}$	$\varepsilon^{ab}\varepsilon^{ijkl}u_{ia}^{(0,1)}u_{jb}^{(0,1)}u_k^{(1,-1)}$ $\times u_l^{(-1,-1)}=1$ and its conjugate	$u_i^{(1,0,1)}u_j^{(-1,0,1)}u_k^{(0,1,-1)}$ $\times u_l^{(0,-1,-1)}\varepsilon^{ijkl}=1$ and its conjugate
Harmonic derivatives preserving unitarity and unimodularity	$D^{(4+)a}=u^{(3+)i}\,\partial/\partial u_a^{i(-)}$ $+u_i^{a(+)}\,\partial/\partial u_a^{i(3-)}$ $D_a^{(4-)}=u_i^{(3-)}\,\partial/\partial u_i^{a(+)}$ $+u_a^{(-)i}\,\partial/\partial u^{(3+)i}$	$D_{ap}^{(2+)}=u_{ai}^{(+)}\,\partial/\partial u_i^{p(-)}$ $-u_p^{i(+)}\,\partial/\partial u^{ai(-)}$ $D_{ap}^{(2-)}=u_a^{i(-)}\,\partial/\partial u^{(+)pi}$ $-u_{pi}^{(-)}\,\partial/\partial u_i^{a(-)}$	$D_a^{(1,2)}=u_{ai}^{(0,1)}\,\partial/\partial u_i^{(-1,-1)}$ $-u^{(1,1)i}\,\partial/\partial u^{ai(0,-1)}$ $D_a^{(1,-2)}=u_a^{i(0,-1)}\,\partial/\partial u^{(-1,1)i}$ $-u_i^{(1,-1)}\partial/\partial u_i^{a(0,1)}$ $D_a^{(-1,2)}=u_{ai}^{(0,1)}\,\partial/\partial u_i^{(1,-1)}$ $-u^{(-1,1)i}\,\partial/\partial u^{(0,-1)ai}$ $D^{(2,0)}=u_i^{(1,-1)}\,\partial/\partial u_i^{(-1,-1)}$ $-u^{(1,1)i}\,\partial/\partial u^{(-1,1)i}$ $D^{(-1,-2)a}=\overline{D_a^{(1,2)}}$ $D^{(-1,2)a}=\overline{D_a^{(1,-2)}}$ $D^{(-2,0)}=\overline{D^{(2,0)}}$	$D^{(1,1,2)}=u_i^{(1,0,1)}\,\partial/\partial u_i^{(0,-1,-1)}$ $-u^{(0,1,1)i}\,\partial/\partial u^{(-1,0,-1)i}$ $D^{(-1,1,2)}=u_i^{(-1,0,1)}\,\partial/\partial u_i^{(0,-1,-1)}$ $-u^{(0,1,1)i}\,\partial/\partial u^{(1,0,-1)i}$ $D^{(1,-1,2)}=u_i^{(1,0,1)}\,\partial/\partial u_i^{(0,1,-1)}$ $-u^{(0,-1,1)i}\,\partial/\partial u^{(-1,0,-1)i}$ $D^{(-1,-1,2)}=u_i^{(-1,0,1)}\,\partial/\partial u_i^{(0,1,-1)}$ $-u^{(0,-1,1)i}\,\partial/\partial u^{(1,0,-1)i}$ $D^{(0,2,0)}=u_i^{(0,1,-1)}\,\partial/\partial u^{(0,-1,-1)}$ $-u^{(0,1,1)i}\,\partial/\partial u^{(0,-1,1)i}$ $D^{(2,0,0)}=-u_i^{(1,0,1)}\partial/\partial u_i^{(-1,0,1)}$ $+u^{(1,0,-1)i}\,\partial/\partial u^{(-1,0,-1)i}$
Converting indices	$\psi_i=u_i^{(+)a}\psi_a^{(-)}-u_i^{(3-)}\psi^{(3+)}$ $\psi_a^{(-)}=u_a^{(-)i}\psi_i$ $\psi^{(3+)}=u^{(3+)i}\psi_i$	$\psi_i=u_i^{(+)a}\psi_a^{(-)}-u_i^{(-)p}\psi_p^{(+)}$ $\psi_a^{(-)}=u_a^{(-)i}\psi_i$ $\psi_p^{(+)}=u_p^{(+)i}\psi_i$	$\psi_i=-u_{ia}^{(0,1)}\psi^{a(0,-1)}$ $+u_i^{(1,-1)}\psi^{(-1,1)}+u_i^{(-1,-1)}\psi^{(1,1)}$ $\psi^{a(0,-1)}=u^{ia(0,-1)}\psi_i$ $\psi^{(-1,1)}=u^{(-1,1)i}\psi_i$ $\psi^{(1,1)}=u^{(1,1)i}\psi_i$	$\psi_i=u_i^{(1,0,1)}\psi^{(-1,0,-1)}$ $+u_i^{(1,0,-1)}\psi^{(-1,0,1)}$ $+u_i^{(0,-1,-1)}\psi^{(0,1,1)}$ $\psi^{(\alpha\beta\gamma)}=\psi_i u^{(\alpha,\beta,\gamma)i}$
Reality and involution (*) (if exists)	No	Yes $u_{ia}^{(+)}\overset{*}{\leftrightarrow}u_{ip}^{(-)}$	No	Yes $u_i^{(1,0,1)}\overset{*}{\leftrightarrow}u_i^{(0,-1,-1)}$ $u_i^{(-1,0,1)}\overset{*}{\leftrightarrow}u_i^{(0,1,-1)}$ $u_i^{(0,1,-1)}\overset{*}{\leftrightarrow}u_i^{(-1,0,1)}$ $u_i^{(0,-1,-1)}\overset{*}{\leftrightarrow}u_i^{(1,0,1)}$

Table 3.

G = USp(2)	H = $SU_1(2)\times SU_2(2)$	H = $SU(2)\times U(1)$	H = $U_1(1)\times U_2(1)$
Generators of H in the 4×4 matrix form	$SU_1(2)$ $\quad T_a=\begin{pmatrix}\tau_a & 0\\ 0 & 0\end{pmatrix}$ $SU_2(2)$ $\quad T_p=\begin{pmatrix}0 & 0\\ 0 & \tau_p\end{pmatrix}$ where τ_a, τ_p are 2×2 $SU_1(2)$ and $SU_2(2)$ algebra's generators	SU(2) $\quad T_a=\begin{pmatrix}\tau_a & 0\\ 0 & 0\end{pmatrix}$ U(1) $\quad T=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$	$U_1(1)$ $\quad T_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$ $U_2(1)$ $\quad T_2=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$
q_i^A	$q_i^a=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $q_i^p=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ a, b, c are doublet indices of $SU_1(2)$ p, q, r are doublet indices of $SU_2(2)$	$q_i^a=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ is SU(2) doublet $q_i^+=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ is SU(2) singlet $q_i^-=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ is SU(2) singlet	$q_i^{(1,0)}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $q_i^{(-1,0)}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $q_i^{(0,1)}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $q_i^{(0,-1)}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Harmonics	u_i^a, u_i^p and their conjugates u_a^i, u_p^i satisfied $u^{ia}=\Omega^{ij}u_j^a$ $u^{ip}=\Omega^{ji}u_i^p$	u_i^a, u_i^+, u_i^- and their conjugates u_a^i, u^{-i}, u^{+i} satisfy $u_i^-=\Omega_{ij}u^{-j}$ $u_i^+=\Omega_{ij}u^{+j}$ $u_i^a=\Omega_{ij}u^{aj}$	$u_i^{(1,0)}$, $u_i^{(-1,0)}$, $u_i^{(0,1)}$, $u_i^{(0,-1)}$ and their conjugates $u^{i(-1,0)}=\Omega^{ij}u_j^{(-1,0)}$ $u^{i(0,-1)}=\Omega^{ij}u_j^{(0,-1)}$ $u^{i(1,0)}=\Omega^{ij}u_j^{(1,0)}$ $u^{i(0,1)}=\Omega^{ij}u_j^{(0,1)}$
Unitarity $uu^+=u^+u=1$	$\Omega^{ij}u_{ip}u_{jq}=\varepsilon_{pq}$ $\Omega^{ij}u_{ia}u_{jb}=\varepsilon_{ab}$ $u_{ia}u_{jb}\varepsilon^{ab}+u_{ip}u_{jq}\varepsilon^{pq}$ $=\Omega_{ij}$	$\Omega^{ij}u_{ai}u_{bj}=\varepsilon_{ab}$ $\Omega^{ij}u_i^+u_j^-=1$ $\Omega^{ij}u_i^+u_j^+=\Omega^{ij}u_i^-u_j^-=0$ $-\varepsilon^{ab}u_{ai}u_{bj}+u_i^+u_j^-$ $-u_j^+u_i^-=-\Omega_{ij}$	$u_i^{(1,0)}u^{i(-1,0)}$ $=u_i^{(0,1)}u^{i(0,-1)}=1$ $u_i^{(1,0)}u^{i(0,1)}=u_i^{(1,0)}u^{i(0,-1)}=0$ $u_i^{(1,0)}u_j^{(-1,0)}-u_j^{(1,0)}u_i^{(-1,0)}$ $+u_i^{(0,1)}u_j^{(0,-1)}-u_j^{(0,1)}u_i^{(0,-1)}$ $=\Omega_{ij}$
Unimodularity $\det u=1$	$\varepsilon^{ijkl}u_{ia}u_{jb}u_{kp}u_{lq}\varepsilon^{ab}\varepsilon^{pq}=1$	$\varepsilon^{ab}u_{ai}u_{bj}u_k^+u_l^-\varepsilon^{ijkl}=1$	$\varepsilon^{ijkl}u_i^{(1,0)}u_j^{(-1,0)}u_k^{(0,1)}u_l^{(0,-1)}=1$
Harmonic derivatives preserving unitarity and unimodularity	$D_{pa}=u_{ia}\,\partial/\partial u_i^p-u_{ip}\,\partial/\partial u_i^a$	$D_a^+=u_{ia}\,\partial/\partial u_i^--u_i^+\,\partial/\partial u_i^a$ $D_a^-=u_{ia}\,\partial/\partial u_i^+-u_i^-\,\partial/\partial u_i^a$ $D^{++}=u_i^+\,\partial/\partial u_i^-$ $D^{--}=u_i^-\,\partial/\partial u_i^+$	$D^{(2,0)}=u_i^{(1,0)}\,\partial/\partial u_i^{(-1,0)}$ $D^{(-2,0)}=u_i^{(-1,0)}\,\partial/\partial u_i^{(1,0)}$ $D^{(0,2)}=u_i^{(0,1)}\,\partial/\partial u_i^{(0,-1)}$ $D^{(0,-2)}=u_i^{(0,-1)}\,\partial/\partial u_i^{(0,1)}$ $D^{(1,1)}=u_i^{(1,0)}\,\partial/\partial u_i^{(0,-1)}$ $-u_i^{(0,1)}\,\partial/\partial u_i^{(-1,0)}$ $D^{(1,-1)}=u_i^{(1,0)}\,\partial/\partial u_i^{(0,1)}$ $-u_i^{(0,-1)}\,\partial/\partial u_i^{(-1,0)}$ $D^{(-1,1)}=\overline{D^{(1,-1)}}$ $D^{(-1,-1)}=\overline{D^{(1,1)}}$
Converting indices	$\psi_i=-u_i^a\psi_a-u_i^p\psi_p$ $\psi_a=u_a^i\psi_i$ $\psi_p=u_p^i\psi_i$	$\psi_i=u_i^a\psi_a-u_i^+\psi^-+u_i^-\psi^+$ $\psi_a=u_a^i\psi_i$ $\psi^\pm=u^{\pm i}\psi_i$ $\psi^i=\Omega^{ij}\psi_j$	$\psi_i=u_i^{(1,0)}\psi^{(-1,0)}-u_i^{(-1,0)}\psi^{(1,0)}$ $+u_i^{(0,1)}\psi^{(0,-1)}-u_i^{(0,-1)}\psi^{(0,1)}$ $\psi^{(\pm1,0)}=-u^{(\pm1,0)i}\psi_i$ $\psi^{(0,\pm1)}=-u^{(0,\pm1)i}\psi_i$
Involution	$u_{ia}\leftrightarrow u_{ip}$	no	$u_i^{(\pm1,0)}\leftrightarrow u_i^{(0,\pm1)}$

Note that in the case of H = SU(2) × U(1) there is no proper involution between u_i^A harmonics allowing us to define an appropriate notion of reality for the harmonic decomposition.

On the other hand, in the case of H = U(1) × U(1) along with the usual complex conjugation

$$u_i^{(1,1)} \leftrightarrow u^{(-1,-1)i}$$

there exists another involution generalising the operation (2.11)

$$u_i^{(1,1)} \overset{*}{\leftrightarrow} u_i^{(0,-2)}$$
$$u_i^{(-1,1)} \leftrightarrow -u_i^{(-1,1)}.$$

Together with the usual conjugation, the latter involution provides a tool for defining reality in the $N=3$ SYM theory (Galperin *et al* 1985).

4. Harmonics for G = SU(4)

In table 2 we briefly list all the results in our analysis with G = SU(4) for H = SU(3) × U(1), H = SU(2) × SU(2) × U(1), H = SU(2) × U(1) × U(1) and H = U(1) × U(1) × U(1).

Note that for the case H = SU(2) × SU(2) × U(1) the involution * is defined as

$$u_{ia}^{(+)} \overset{*}{\leftrightarrow} u_{ip}^{(-)}$$

which together with the usual conjugation

$$\bar{u}_{ia}^{(+)} = u^{ia(-)} \qquad \bar{u}_{ip}^{(-)} = u^{ip(+)}$$

allows us to define reality in $N=4$ SUSY theory. Reality can be imposed also with H = U(1) × U(1) × U(1) due to the existence of the involution

$$u_i^{(1,0,1)} \overset{*}{\leftrightarrow} u_i^{(0,-1,-1)} \qquad u_i^{(0,1,-1)} \overset{*}{\leftrightarrow} u_i^{(-1,0,1)}$$
$$u_i^{(-1,0,1)} \leftrightarrow u_i^{(0,1,-1)} \qquad u_i^{(0,-1,-1)} \leftrightarrow u_i^{(1,0,1)}$$

(the other forms of the latter are possible as well).

5. Harmonics for G = USp(2)

Before establishing the harmonic analysis for G = USp(2) let us briefly recall the definition and introduce the matrix representation of USp(2).

The Lie algebra corresponding to USp(2) is formed by 2 × 2 quaternionic matrices

$$\begin{pmatrix} a & c \\ -\bar{c} & b \end{pmatrix} \in \mathrm{USp}(2) \tag{5.1}$$

a, b are pure imaginary quaternions ($\bar{a} = -a$, $\bar{b} = -b$), c is an arbitrary quaternion.

Representing a and b as SU(2) matrices, we obtain the following 4×4 matrix representation

$$\left(\begin{array}{c|c} SU_1(2) & c \\ \hline -\bar{c} & SU_2(2) \end{array}\right) \in USp(2). \tag{5.2}$$

c is a general GL(2, R) matrix. Matrices belonging to the USp(2) group are unitary and have unit determinant as can be checked easily from (5.1) and (5.2). USp(2) is a subgroup of SU(4). The USp(2) group is also known to possess an invariant antisymmetric tensor of second rank Ω_{ij} (symplectic metrics), that is

$$\Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.3}$$

We shall list below the harmonic analysis on USp(2) cosets with $H = SU_1(2) \times SU_2(2)$; $H = SU_1(2) \times U_2(1)$; $H = U_1(1) \times U_2(1)$, $U_I(1)$ being the diagonal subgroups of $SU_I(2)$, $I = 1, 2$ (table 3).

6. Conclusion

So far we have constructed objects that can connect the representation space of some group with the representation space of some of its subgroups. This situation will be especially useful in N-extended SYM theories. In those theories, by adding the set of harmonics, we come to the concept of harmonic superspace. It turns out that in the harmonic superspace there is a hypersurface spanned by the so-called analytic basis. All the fundamental objects (e.g. superfields) appear naturally in this basis. Roughly speaking, the appearance of analytic basis is connected with the existence of the Cauchy-Riemann (CR) structure (Rosly 1982). With the help of harmonics SU(N) indices of the spinor covariant derivatives fall into the $H \subset SU(N)$ ones. After the Yang-Mills covariantisation of those spinor derivatives by adding connections to them, we can pick out a subset fulfilling the 'flat' algebra. This condition is equivalent to the integrability condition of the Cauchy-Riemann structure. Its presence crucially depends on the choice of H. For the $N = 3$ case, e.g., $H = SU(2) \times U(1)$ does not allow the existence of CR structure as $H = U(1) \times U(1)$ does. In the case $N = 4$, CR structure does not exist with $H = SU(3) \times U(1)$, $SU(2) \times SU(2) \times U(1)$, $SU(2) \times U(1) \times U(1)$ while it can be constructed for $H = U(1) \times U(1) \times U(1)$.

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